An axiomatic theory for anonymized risk sharing

Zhanyi Jiao

Department of Statistics and Actuarial Science
University of Waterloo

Waterloo Student Conference

October 14, 2022

What this talk is about

- Establish an axiomatic framework for anonymized risk sharing.
- Require NO information on preferences, identities, private operations and realized losses from individual agents.
- Identify four axioms: actuarial fairness, risk fairness, risk anonymity and operational anonymity.
- Characterize the conditional mean risk sharing (CMRS) rule.
- Provide an application on Bitcoin mining pools.

Based on joint work with Steven Kou (Boston), Yang Liu (Stanford) and Ruodu Wang (Waterloo)

UNIVERSITY OF WATERLOO

Background and Motivation

"Risk sharing" - Section 1

- Collective risk sharing
  - Deriving Pareto equilibrium
  - Requiring central planner who knows preferences of all agents
- Competitive risk sharing
  - Deriving competitive equilibrium
  - Requiring trading mechanism and individual preferences

"Anonymized" - Section 2

- No central planner and preference
- No private operations
- No individual realized losses

"Axiomatic theory" - Section 3

- Axiomatization for decision theory (e.g. Yaari (1987))
- Axiomatization for risk measure (e.g. Artzner et. al (1999))

UNIVERSITY OF WATERLOO

Contents

UNIVERSITY OF WATERLOO

Preliminary

- $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
- $\mathcal{X}$ be a space of random variables (e.g., $\mathcal{X} = L^1$).
- n economic agents share a total risk, where $n \geqslant 3$ is an integer.
- Each agent $i \in [n] = \{1, \ldots, n\}$ faces an initial risk $X_i \in \mathcal{X}$.
- Let $\mathbf{X} = (X_1, \ldots, X_n)$ be the initial risk (contribution) vector, and $S^{\mathbf{X}} = \sum_{i=1}^n X_i$ be the total risk.
- For any random variable S, the set of all allocations of S is denoted by

$$\mathbb{A}_n(S) = \left\{ (Y_1, \ldots, Y_n) \in \mathcal{X}^n : \sum_{i=1}^{n} Y_i = S \right\}.$$

UNIVERSITY OF WATERLOO

Risk sharing

Definition 1: Risk sharing

A risk sharing rule is a mapping $\mathbf{A} : \mathcal{X}^n \to \mathcal{X}^n$ satisfying

$$\mathbf{A}^{\mathbf{X}} = (A_1^{\mathbf{X}}, \dots, A_n^{\mathbf{X}}) \in \mathbb{A}_n(S^{\mathbf{X}})$$

for each $\mathbf{X} \in \mathcal{X}^n$.

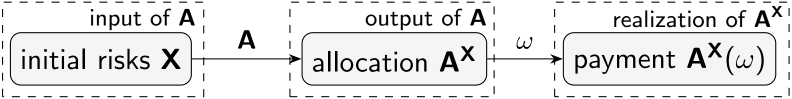

Figure: Risk sharing procedure

Examples (assume $\mathcal{X} \in L^1$)

- The identity risk sharing rule

$$\mathbf{A}^{\mathbf{X}}_{\mathrm{id}} = \mathbf{X}.$$

- The all-in-one risk sharing rule

$$\mathbf{A}^{\mathbf{X}}_{\mathrm{all}} = \left(S^{\mathbf{X}}, 0, \ldots, 0 \right).$$

- The mean proportional risk sharing rule

$$\mathbf{A}^{\mathbf{X}}_{\mathrm{prop}} = \frac{S^{\mathbf{X}}}{\mathbb{E}[S^{\mathbf{X}}]} \mathbb{E}[\mathbf{X}].$$

- The uniform risk sharing rule

$$\mathbf{A}^{\mathbf{X}}_{\mathrm{unif}} = S^{\mathbf{X}} \left(\frac{1}{n}, \ldots, \frac{1}{n} \right).$$

- The conditional mean risk sharing rule (CMRS)

$$\mathbf{A}^{\mathbf{X}}_{\mathrm{cm}} = \mathbb{E}\left[\mathbf{X} | S^{\mathbf{X}} \right].$$

UNIVERSITY OF WATERLOO

Contents

UNIVERSITY OF WATERLOO

Actuarial Fairness

Axiom AF (Actuarial fairness)

The expected value of each agent's allocation coincides with the expected value of the initial risk, that is

$$\mathbb{E}[\mathbf{A}^{\mathbf{X}}] = \mathbb{E}[\mathbf{X}] \quad \text{for} \quad \mathbf{X} \in \mathcal{X}^n.$$

- Axiom AF serves as the basis for premium pricing in insurance
- Axiom AF is natural for anonymized framework since no information on the preferences or identities of the agents

UNIVERSITY OF WATERLOO

Risk Fairness

Axiom RF (Risk fairness)

The allocation to each agent should not exceed their maximum possible loss. That is, for $\mathbf{X} \in \mathcal{X}^n$ and $i \in [n]$, it holds that

$$A_i^{\mathbf{X}} \leqslant \sup X_i.$$

- Can be formulated as $A_i^{\mathbf{X}} \geqslant \inf X_i$.
- Pure surplus ($X_i \leqslant 0$) leads to pure surplus allocation.
- Axiom AF and RF $\Longrightarrow$ $X_i = c$ is a constant, then $A_i^{\mathbf{X}} = c$.

  Example: For initial risk vector $(X, 0, \ldots, 0)$, AF+RF leads to

$$A_1^{(X,0,\ldots,0)} = X \text{ and } A_j^{(X,0,\ldots,0)} = 0 \text{ for } j \neq 1.$$

Risk Anonymity

Axiom RA (Risk anonymity)

The realized value of the allocation to each agent is determined by that of the total risk. That is, for $\mathbf{X} \in \mathcal{X}^n$,

$$\mathbf{A}^{\mathbf{X}} \text{ is } \sigma(S^{\mathbf{X}}) - \textit{measurable}.$$

- All individual losses are masked and ONLY the total loss is revealed
- Initial risk vector is only used for the design of the risk sharing mechanism, but NOT for the settlement of actual losses (see e.g., Figure 1).

UNIVERSITY OF WATERLOO

Operational Anonymity

Axiom OA (Operational anonymity)

The allocation to one agent is not affected if risks of two other agents merge. That is,

$$\mathbf{Y} = \mathbf{X} + X_j\mathbf{e}_i - X_j\mathbf{e}_j \Longrightarrow A_k^{\mathbf{Y}} = A_k^{\mathbf{X}} \text{ for } k \neq i, j$$

where $\mathbf{e}_k = (0, \cdots, 0, 1, 0, \cdots, 0)$ is the unit vector along the k-th axis (the k-th component is 1).

- Example: $\mathbf{X} = (X_1, X_2, \cdots, X_n)$, $\mathbf{Y} = (X_1 + X_2, 0, \cdots, X_n) \Longrightarrow A_k^{\mathbf{Y}} = A_k^{\mathbf{X}}$ for $k \neq 1, 2$.
- Two agent may be two different accounts of same person or family.
- Internal operations do not need to be disclosed and affect the allocation to other agents.

UNIVERSITY OF WATERLOO

Contents

Conditional mean risk sharing (CMRS)

Definition 2 (Conditional mean risk sharing)

The conditional mean risk sharing rule is defined as

$$\mathbf{A}^{\mathbf{X}} = \mathbb{E}\left[\mathbf{X}|S^{\mathbf{X}}\right] \quad \text{for } \mathbf{X} \in \mathcal{X}^n \subseteq (L^1)^n,$$

and equivalently, $A_i^{\mathbf{X}} = \mathbb{E}[X_i|S^{\mathbf{X}}]$ for $i \in [n]$

- CMRS was used by Landsberger and Meilijson (1994), but named and studied in detail by Denuit and Dhaene (2012).
- A popular risk sharing rule in decentralized system
  - P2P insurance (Denuit and Robert (2021), Feng et al. (2021,2022))
  - Bitcoin mining (Jiao et al. (2022))
  - Tontines (Hieber and Lucas (2022))

UNIVERSITY OF WATERLOO

Axiomatic characterization of CMRS

Theorem 1

Assume $\mathcal{X} = L^1$, i.e., the set of all integrable random variables. A risk sharing rule satisfies Axioms AF, RF, RA and OA if and only if it is CMRS.

Theorem 2

Assume $\mathcal{X} = L^1_+$, i.e., the set of all non-negative integrable random variables. A risk sharing rule satisfies Axioms AF, RF, RA and OA if and only if it is CMRS.

UNIVERSITY OF WATERLOO

A new characterization of conditional expectation

Theorem 3

For a random variable S on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} = \sigma(S)$, let $\phi : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to L^1(\Omega, \mathcal{G}, \mathbb{P})$. The equality $\phi(X) = \mathbb{E}[X|S]$ holds for all $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ if and only if ϕ satisfies the following properties:

(a) $\phi(t) = t$ for all $t \in \mathbb{R}$;

(b) $\phi(X + Y) = \phi(X) + \phi(Y)$ for all X, Y;

(c) $\phi(Y) \geqslant \phi(X)$ if $Y \geqslant X$;

(d) $\phi(S) = S$;

(e) $\mathbb{E}[\phi(X)] = \mathbb{E}[X]$ for all X.

Independence of axioms

Proposition 1

Axioms AF, RF, RA and OA are independent. Any combination of three of Axioms AF, RF, RA and OA does not imply the remaining fourth axiom.

- RF, RA and OA, but NOT AF:

$$\mathbf{A}^{\mathbf{X}}_{Q\text{-cm}} = \mathbb{E}^{Q}[\mathbf{X}|S^{\mathbf{X}}] \text{ for } Q \neq \mathbb{P}$$

- AF, RA and OA, but NOT RF:

$$\mathbf{A}^{\mathbf{X}}_{\mathrm{ma}} = \left(S^{\mathbf{X}} - \mathbb{E}[S^{\mathbf{X}}], 0, \dots, 0\right) + \mathbb{E}[\mathbf{X}]$$

- AF, RF and OA, but NOT RA:

$$\mathbf{A}^{\mathbf{X}}_{\mathrm{id}} = \mathbf{X}$$

- AF, RF and RA, but NOT OA:

$$\mathbf{A}^{\mathbf{X}} = \mathbf{A}^{\mathbf{X}}_{\mathrm{all}} \mathbb{1}_B + \mathbf{A}^{\mathbf{X}}_{\mathrm{cm}} \mathbb{1}_{B^c}$$

with $B = \{\mathbf{X} \text{ is standard Gaussian }\}$

Contents

UNIVERSITY OF WATERLOO

An application: Bitcoin mining pool

- n miners form a mining pool and share the possible reward
- Miner all possible contribution vector:
  $\mathcal{B}_n = \{P(\mathbb{1}_{D_1}, \dots, \mathbb{1}_{D_n}) : D_1, \dots, D_n \text{ disjoint and independent of } P\}$.
  - $P > 0$ (rv): monetary value of next block.
  - D_i: event that miner i solves the next block.
  - $D_1, \dots, D_n$ mutually exclusive and $D = \bigcup_{i=1}^n D_i$.
  - $\mathbb{P}(D_i)$: computational contribution of the miner i.
- Rewarding sharing rule $\mathbf{A} : \mathcal{B}_n \to \mathcal{X}^n$ satisfying
  - $\mathbf{A}^{\mathbf{X}} = \mathbb{A}_n(S^{\mathbf{X}})$ for each $\mathbf{X} \in \mathcal{B}_n$
  - $A_i^{\mathbf{X}} = A_j^{\mathbf{X}}$ for $i, j \in [n]$ with $\mathbb{P}(D_i) = \mathbb{P}(D_j)$ where $\mathbf{X} = P(\mathbb{1}_{D_1}, \dots, \mathbb{1}_{D_n})$

Rewarding sharing rule

Proposition 3

Assume $P \in \mathcal{X} = L^1$ and $P > 0$. A reward sharing rule $\mathbf{A} : \mathcal{B}_n \to \mathcal{X}^n$ satisfies Axioms RA, RF, AF and OA if and only if it is specified by

$$A_i^{\mathbf{X}} = \frac{\mathbb{P}(D_i)}{\mathbb{P}(D)} P \mathbb{1}_D, \quad i \in [n], \ \mathbf{X} = P(\mathbb{1}_{D_1}, \ldots, \mathbb{1}_{D_n}) \in \mathcal{B}_n, \tag{1}$$

which is CMRS ($\mathbb{E}[P\mathbb{1}_{D_i} | P\mathbb{1}_D] = P\mathbb{1}_D \mathbb{P}(D_i)/\mathbb{P}(D)$.).

- AF: No miner gets less (or more) than initial contribution in expectation.
- RF ($A_i^{\mathbf{X}} \geqslant \inf X_i$): Any miner has a non-negative reward
- RA: Reward does not depend on which miner issues the block
- OA: Mechanism safe against merging

An example: A pool of three miner

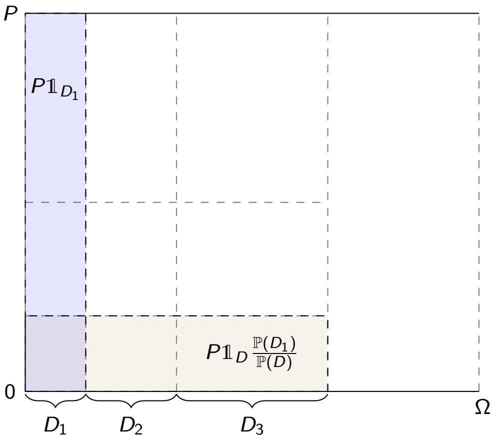

Figure: Bitcoin mining pool of 3 miners

UNIVERSITY OF WATERLOO

Conclusion

- New axiomatic framework for anonymized risk sharing
  - No central planner
  - No preference and identities revealed
  - No private operation revealed
  - No individual realized losses revealed
- Identify four natural economic axioms
  - Axiom AF and RF reflect fairness;
  - Axiom OA reflects irrelevance of unseen operations;
  - Axiom RA reflects no individual realized loss needs to disclose.
- Axiom AF, RF, OA and RA $\Longleftrightarrow$ CMRS
- An application on Bitcoin mining pool

Reference

Denuit, M. and Dhaene, J. (2012)
Convex order and comonotonic conditional mean risk sharing
Insurance: Mathematics and Economics 51(2), 265–270.

Denuit, M., Dhaene, J. and Robert, C. Y (2022)
Risk-sharing rules and their properties, with applications to peer-to-peer insurance
Journal of Risk and Insurance 89(3), 615–667

Denuit, M. and Robert, C. Y. (2020)
Risk reduction by conditional mean risk sharing with application to collaborative insurance.
Available at https://ideas.repec.org/p/aiz/louvad/2020024.html

Jiao, Z., Liu, Y., and Wang, R. (2022).
An axiomatic theory for anonymized risk sharing.
arXiv preprint arXiv:2208.07533

Landsberger, M. and Meilijson, I. (1994)
Co-monotone allocations, Bickel-Lehmann dispersion and the Arrow-Pratt measure of risk aversion
Annals of Operations Research 52(2), 97–106.

Leshno, J. D. and Strack, P. (2020)
Bitcoin: An axiomatic approach and an impossibility theorem.
American Economic Review: Insights 2(3), 269–286.

Thank you for your attention!