Worst-case upper partial moment risk measures with application to robust portfolio selection

Zhanyi Jiao

Department of Statistics and Actuarial Science
University of Waterloo

26th International Congress on IME

July 7, 2023

What this talk is about

- Revisit worst-case first-order upper partial moment (UPM) under uncertainty set induced by mean and variance, and subsets with additional conditions including symmetrical distribution, non-negative random loss.
- Derive closed-form worst-case second-order UPM (target semi-variance) under different uncertainty sets.
- Develop worst-case target semi-variance with constraints on expected losses over target levels (first-order UPM).
- Provide applications on robust portfolio selection with different objectives.

Based on joint work with Jun Cai (Waterloo) and Tiantian Mao (USTC).

Contents

Upper partial moment (UPM) risk measures

Let X be the random loss and $F \in \mathcal{F}$ be the distribution of X.

Definition 1 (Upper partial moment)

Given a threshold level $t \in \mathbb{R}$, the n-th order UPM of X is defined as

$$\mathbb{E}^F[(X-t)_+^n] = \int_t^\infty (x-t)^n \, \mathrm{d}F(x).$$

- **Finance**: Allows investors to set a subjective target for the perceived level of investment risk to measure the downside risk. (Chen/He/Zhang'11, Bertsimas/Popescu'02)

  Target semi-variance $\mathbb{E}^F[(X-t)_+^2]$ *v.s* $\mathbb{E}^F[(X-\mathbb{E}^F[X])_+^2]$
- **Insurance**: Stop-loss premium principle, semi-variance premium principle. (Kaluszka'05, Cai/Tan'07, Cai/Chi'20)
- **Economic**: Connection to stochastic dominance and expected utility theory. (Bawa'75, Gomez/Tang/Tong'22)

Worst-case risk under model uncertainty

- Classical models often assume complete knowledge of the loss distribution.
- Gap between the true distribution and the underlying distribution due to insufficient data, prediction errors, or incorrect judgments.
$\Rightarrow$ (distributional) model uncertainty
- Consider the worst-case scenario given partial information of the underlying distribution as a compensation
  - Finance (portfolio selection): Ben-Tal/Nemirovski'98, Chen/He/Zhang'11, Liu/Yang/Yu'21.
  - Insurance: Liu/Mao'22, Cai/Liu/Yin'23.

Uncertainty sets

The general model uncertainty problem with UPM risk measures is formulated as follows:

$$\sup_{F \in \mathcal{L}} \int_t^{\infty} (x-t)^n \, \mathrm{d}F(x).$$

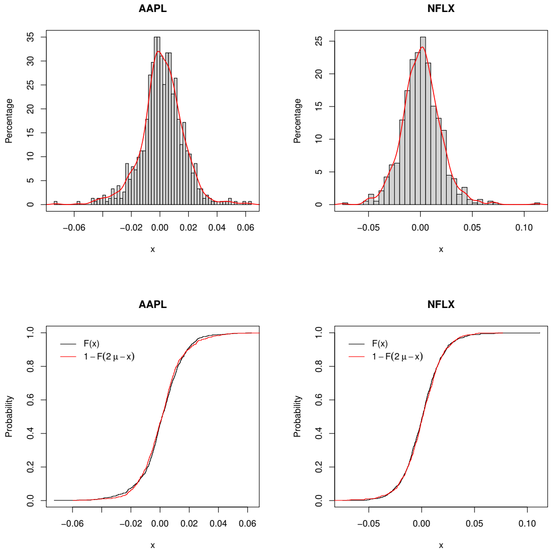
Given the mean μ and the standard deviation σ, the uncertainty set $\mathcal{L}(\mu, \sigma)$ induced by first two moments, and its subsets are denoted by

$$\mathcal{L}(\mu,\sigma) = \left\{ F \in \mathcal{F} : \int_{-\infty}^{\infty} x \, \mathrm{d}F(x) = \mu, \int_{-\infty}^{\infty} x^2 \, \mathrm{d}F(x) = \mu^2 + \sigma^2 \right\},$$

$$\mathcal{L}_+(\mu,\sigma) = \left\{ F \in \mathcal{F} : \int_{0}^{\infty} x \, \mathrm{d}F(x) = \mu, \int_{0}^{\infty} x^2 \, \mathrm{d}F(x) = \mu^2 + \sigma^2, \; F(0-) = 0 \right\},$$

$$\mathcal{L}_S(\mu,\sigma) = \left\{ F \in \mathcal{F} : \int_{-\infty}^{\infty} x \, \mathrm{d}F(x) = \mu, \int_{-\infty}^{\infty} x^2 \, \mathrm{d}F(x) = \mu^2 + \sigma^2, \; F \text{ is symmetric} \right\}.$$

Symmetrical distribution and uncertainty set



Contents

Uncertainty sets reduction techniques

Lemma 1

For any $t \in \mathbb{R}$, we have

$$\sup_{F \in \mathcal{L}(\mu,\sigma)} \mathbb{E}^F[(X-t)_+] = \sup_{F \in \mathcal{L}_2(\mu,\sigma)} \mathbb{E}^F[(X-t)_+].$$

- $\mathcal{L}_k(\mu,\sigma) = \{F \in \mathcal{L}(\mu,\sigma) : F \text{ is a } k\text{-point distribution}\}$.
- Since $\mathcal{L}_2(\mu,\sigma) \subset \mathcal{L}(\mu,\sigma)$, we have

$$\sup_{F \in \mathcal{L}(\mu,\sigma)} \mathbb{E}^F[(X-t)_+] \geq \sup_{F \in \mathcal{L}_2(\mu,\sigma)} \mathbb{E}^F[(X-t)_+].$$

- Construct a two-point rv. X_ϵ

$$X_\epsilon = (\mathbb{E}^F[X|X>t] + \epsilon p)1_{\{X \leq \mu\}} + (\mathbb{E}^F[X|X \leq t] - \epsilon q)1_{\{\mu < X \leq t\}},$$

prove that there exist a two-point distribution such that

$$\sup_{F \in \mathcal{L}(\mu,\sigma)} \mathbb{E}^F[(X-t)_+] \leq \sup_{F \in \mathcal{L}_2(\mu,\sigma)} \mathbb{E}^F[(X-t)_+].$$

Uncertainty sets reduction techniques (cont')

Lemma 2

For any $t \in \mathbb{R}$, we have

$$\sup_{F \in \mathcal{L}_S(\mu,\sigma)} \mathbb{E}^F[(X-t)_+] = \sup_{F \in \mathcal{L}_{3,S}(\mu,\sigma)} \mathbb{E}^F[(X-t)_+].$$

Lemma 3

For any $t \in \mathbb{R}$, we have

$$\sup_{F \in \mathcal{L}_+(\mu,\sigma)} \mathbb{E}^F[(X-t)_+] = \sup_{F \in \mathcal{L}_{+3}(\mu,\sigma)} \mathbb{E}^F[(X-t)_+].$$

Note that, for $k = 2, 3, \dots$, we also define

$$\mathcal{L}_{k,S}(\mu,\sigma) = \{F \in \mathcal{L}_S(\mu,\sigma) : F \text{ is a } k\text{-point symmetric distribution}\},$$
$$\mathcal{L}_{+k}(\mu,\sigma) = \{F \in \mathcal{L}_+(\mu,\sigma) : F \text{ is a } k\text{-point non-negative distribution}\}.$$

Worst-case first-order UPM

Proposition 1 (Jagannathan'77)

If the uncertainty set of X is $\mathcal{L}(\mu,\sigma)$, then

$$\sup_{F\in\mathcal{L}(\mu,\sigma)} \mathbb{E}^F[(X-t)_+] = \frac{1}{2}\left(\mu - t + \sqrt{\sigma^2+(\mu-t)^2}\right).$$

If the uncertainty set of X is $\mathcal{L}_S(\mu,\sigma)$, then

$$\sup_{F\in\mathcal{L}_S(\mu,\sigma)} \mathbb{E}^F[(X-t)_+] = \begin{cases} \frac{8(\mu-t)^2+\sigma^2}{8(\mu-t)}, & t < \mu - \frac{\sigma}{2}, \\ \frac{1}{2}(\mu+\sigma-t), & \mu-\frac{\sigma}{2} \le t < \mu + \frac{\sigma}{2}, \\ \frac{\sigma^2}{8(t-\mu)}, & t \ge \mu + \frac{\sigma}{2}. \end{cases}$$

If the uncertainty set of X is $\mathcal{L}_+(\mu,\sigma)$ and $\mu > 0$, then

$$\sup_{F\in\mathcal{L}_+(\mu,\sigma)} \mathbb{E}^F[(X-t)_+] = \begin{cases} \mu - t, & t < 0, \\ \mu - \frac{\mu^2 t}{\sigma^2+\mu^2}, & 0 \le t < \frac{\sigma^2+\mu^2}{2\mu}, \\ \frac{1}{2}(\mu - t + \sqrt{\sigma^2+(\mu-t)^2}), & t \ge \frac{\sigma^2+\mu^2}{2\mu}, \end{cases}$$

Contents

Worst-case target semi-variance

Proposition 2

If the uncertainty set of X is $\mathcal{L}(\mu,\sigma)$, then

$$\sup_{F\in\mathcal{L}(\mu,\sigma)} \mathbb{E}^F[(X-t)_+^2] = \begin{cases} \sigma^2+(\mu-t)^2 & t\le\mu, \\ \sigma^2 & t>\mu. \end{cases}$$

If the uncertainty set of X is $\mathcal{L}_+(\mu,\sigma)$ and $\mu>0$, then

$$\sup_{F\in\mathcal{L}_+(\mu,\sigma)} \mathbb{E}^F[(X-t)_+^2] = \sup_{F\in\mathcal{L}(\mu,\sigma)} \mathbb{E}^F[(X-t)_+^2].$$

If the uncertainty set of X is $\mathcal{L}_S(\mu,\sigma)$, then

$$\sup_{F\in\mathcal{L}_S(\mu,\sigma)} \mathbb{E}^F[(X-t)_+^2] = \begin{cases} \sigma^2+(\mu-t)^2, & t\le\mu-\sigma, \\ \frac{\sigma^2+3(t-\mu)^2}{2}, & \mu-\sigma<t\le\mu, \\ \frac{\sigma^2}{2}, & t>\mu. \end{cases}$$

Worst-case target semi-variance with expected regret constraint

We assume the risk budget limit $m \in \mathbb{R}^+$ and consider the following optimization problem:

$$\sup_{F \in \mathcal{L}(\mu,\sigma)} \mathbb{E}^F[(X-t)_+^2],$$
$$s.t. \quad \mathbb{E}^F[(X-t)_+] \le m.$$

which is equivalent to the following optimization problem:

$$\sup_{F \in \mathcal{L}_m(\mu,\sigma)} \mathbb{E}^F[(X-t)_+^2],$$

where

$$\mathcal{L}_m(\mu,\sigma) = \left\{ F \in \mathcal{F} : \int_{-\infty}^{\infty} x \ \mathrm{d}F(x) = \mu, \ \int_{-\infty}^{\infty} x^2 \ \mathrm{d}F(x) = \mu^2 + \sigma^2, \int_t^{\infty} (x-t) \ \mathrm{d}F(x) \le m \right\}.$$

Worst-case target semi-variance with expected regret constraint (cont')

Theorem 1

For $t, \mu \in \mathbb{R}$, $\sigma, m \in \mathbb{R}^+$, assume $m > (t-\mu)_-$. Then

$$\sup_{F \in \mathcal{L}_m(\mu,\sigma)} \mathbb{E}^F[(X-t)_+^2] = \begin{cases} \sigma^2 + (t-\mu)^2, & \mu - m < t < \mu, \\ \sigma^2, & t \geq \mu, \end{cases}$$

Corollary 2

For $t \in \mathbb{R}$ and $\mu, \sigma \in \mathbb{R}^+$, we have

$$\sup_{F \in \mathcal{L}_{+,m}(\mu,\sigma)} \mathbb{E}^F[(X-t)_+^2] = \sup_{F \in \mathcal{L}_m(\mu,\sigma)} \mathbb{E}^F[(X-t)_+^2].$$

Contents

TSV-targeted portfolio selection

- Random loss vector in a portfolio: $\mathbf{X}^\top = (X_1, ..., X_d) \in \mathbb{R}^d$.
- Allocation/selection of portfolio: $\boldsymbol{w} = (w_1, ..., w_d) \in \mathbb{R}^d$.
- Total loss of the portfolio: $\boldsymbol{w}^\top \boldsymbol{X} = w_1 X_1 + \cdots + w_d X_d$

The TSV-targeted robust portfolio optimization formulated as follows:

$$\min_{\boldsymbol{w}\in\mathbb{R}^d} \sup_{G\in\mathcal{M}} \mathbb{E}^G[(\boldsymbol{w}^\top \boldsymbol{X} - t)_+]^2$$

$$s.t. \quad \boldsymbol{w}^\top \boldsymbol{e} = 1.$$

$$\mathcal{M}_S(\boldsymbol{\mu}, \Gamma) = \{G \in \mathcal{G} : \mathbb{E}[\boldsymbol{X}] = \boldsymbol{\mu},\ \mathrm{cov}[\boldsymbol{X}] = \Gamma,\ G \text{ is symmetric}\},$$

$$\mathcal{M}_m(\boldsymbol{\mu}, \Gamma) = \left\{G \in \mathcal{G} : \mathbb{E}[\boldsymbol{X}] = \boldsymbol{\mu},\ \mathrm{cov}[\boldsymbol{X}] = \Gamma,\ \mathbb{E}^G[(\boldsymbol{w}^\top \boldsymbol{X} - t)_+] \le m\right\}.$$

Multi-dimensional sets transformation

Lemma 4

If $\boldsymbol{w} \neq \mathbf{0}$, then the following expressions hold:

(1) $\mathcal{M}_{\boldsymbol{w}}(\boldsymbol{\mu}, \Gamma) = \mathcal{L}(\boldsymbol{w}^\top \boldsymbol{\mu}, \boldsymbol{w}^\top \Gamma \boldsymbol{w})$,

(2) $\mathcal{M}_{\boldsymbol{w},S}(\boldsymbol{\mu}, \Gamma) = \mathcal{L}_S(\boldsymbol{w}^\top \boldsymbol{\mu}, \boldsymbol{w}^\top \Gamma \boldsymbol{w})$,

(3) $\mathcal{M}_{\boldsymbol{w},m}(\boldsymbol{\mu}, \Gamma) = \mathcal{L}_m(\boldsymbol{w}^\top \boldsymbol{\mu}, \boldsymbol{w}^\top \Gamma \boldsymbol{w})$,

where the sets $\mathcal{L}$, $\mathcal{L}_S$ and $\mathcal{L}_m$ are one-dimensional uncertainty sets defined previously.

For random vector $\boldsymbol{X}$ with distribution G, denote the corresponding sets of possible distributions of $\boldsymbol{w}^\top \boldsymbol{X}$

(1) $\mathcal{M}_{\boldsymbol{w}}(\boldsymbol{\mu}, \Gamma) = \left\{F_{\boldsymbol{w}^\top \boldsymbol{X}} \in \mathcal{F} : \ G \in \mathcal{M}(\boldsymbol{\mu}, \Gamma)\right\}$,

(2) $\mathcal{M}_{\boldsymbol{w},S}(\boldsymbol{\mu}, \Gamma) = \left\{F_{\boldsymbol{w}^\top \boldsymbol{X}} \in \mathcal{F} : \ G \in \mathcal{M}_S(\boldsymbol{\mu}, \Gamma)\right\}$,

(3) $\mathcal{M}_{\boldsymbol{w},m}(\boldsymbol{\mu}, \Gamma) = \left\{F_{\boldsymbol{w}^\top \boldsymbol{X}} \in \mathcal{F} : \ G \in \mathcal{M}_m(\boldsymbol{\mu}, \Gamma)\right\}$.

Robust TSV-targeted portfolio selection

The inner part of original multiple-dimensional optimization problem is transformed into one-dimensional problem

$$\min_{\boldsymbol{w}\in\mathbb{R}^d} \sup_{F\in\mathcal{L}(\boldsymbol{w}^\top\boldsymbol{\mu},\boldsymbol{w}^\top\Gamma\boldsymbol{w})} \mathbb{E}^F[(\boldsymbol{w}^\top\boldsymbol{X}-t)_+]^2$$

$$s.t. \quad \boldsymbol{w}^\top\boldsymbol{e}=1.$$

Denote

$$u=(\boldsymbol{e}^\top\Gamma^{-1}\boldsymbol{e})(\boldsymbol{\mu}^\top\Gamma^{-1}\boldsymbol{\mu})-(\boldsymbol{e}^\top\Gamma^{-1}\boldsymbol{\mu})^2,$$

$$v_0=\frac{\boldsymbol{e}^\top\Gamma^{-1}\boldsymbol{e}}{u}, \quad v_1=\frac{\boldsymbol{e}^\top\Gamma^{-1}\boldsymbol{\mu}}{u}, \quad v_2=\frac{\boldsymbol{\mu}^\top\Gamma^{-1}\boldsymbol{\mu}}{u}.$$

Robust TSV-targeted portfolio selection

Proposition 3

Let Γ be a positive definite matrix. For $t \in \mathbb{R}$, $\boldsymbol{w} \in \mathbb{R}^d$ with $\boldsymbol{w}^\top \boldsymbol{e} = 1$, The optimal portfolio selection $\boldsymbol{w}^*$ has the following expressions:

(1) If $\mathcal{M} = \mathcal{M}_S(\boldsymbol{\mu}, \Gamma)$, then

$$\boldsymbol{w}_S^* = (\Gamma^{-1}\boldsymbol{\mu}, \quad \Gamma^{-1}\boldsymbol{e}) \begin{pmatrix} v_0 & -v_1 \\ -v_1 & v_2 \end{pmatrix} \begin{pmatrix} \xi_{S,t}^* \\ 1 \end{pmatrix},$$

where $\xi_{S,t}^* = \arg\min_{\xi \in \mathbb{R}} \ h_{S,t}(\xi, \sqrt{v_0\xi^2 - 2v_1\xi + v_2})$, and $h_{S,t}(\mu, \sigma) = \sup_{F \in \mathcal{L}_S(\mu,\sigma)} \mathbb{E}^F[(X - t)_+^2]$.

(2) If $\mathcal{M} = \mathcal{M}_m(\boldsymbol{\mu}, \Gamma)$, then

$$\boldsymbol{w}_m^* = (\Gamma^{-1}\boldsymbol{\mu}, \quad \Gamma^{-1}\boldsymbol{e}) \begin{pmatrix} v_0 & -v_1 \\ -v_1 & v_2 \end{pmatrix} \begin{pmatrix} \xi_{m,t}^* \\ 1 \end{pmatrix},$$

where $\xi_{m,t}^* = \arg\min_{\xi \in \mathbb{R}} \ h_{m,t}(\xi, \sqrt{v_0\xi^2 - 2v_1\xi + v_2})$, and $h_{m,t}(\mu, \sigma) = \sup_{F \in \mathcal{L}_m(\mu,\sigma)} \mathbb{E}^F[((X - t)_+^2]$.

HMCR-targeted portfolio selection

Classic mean-HMCR portfolio optimization model is formulated as follows:

$$\min_{\boldsymbol{w}\in\mathbb{R}^d} \left\{ \mathbb{E}^G[\boldsymbol{w}^\top \boldsymbol{X}] + \lambda \underbrace{\min_{c\in\mathbb{R}} \left(c + \theta \left(\mathbb{E}^G[(\boldsymbol{w}^\top \boldsymbol{X} - c)_+^p] \right)^{\frac{1}{p}} \right)}_{\text{Higher moment coherent risk (HMCR) measure}} \right\}$$

$$s.t. \quad \boldsymbol{w}^\top \boldsymbol{e} = 1.$$

Robust mean-HMCR portfolio optimization formulated as follows:

$$\min_{(\boldsymbol{w},c)\in\mathbb{R}^d\times\mathbb{R}} \sup_{G\in\mathcal{M}} \left\{ \mathbb{E}^G[\boldsymbol{w}^\top \boldsymbol{X}] + \lambda \left(c + \theta \left(\mathbb{E}^G[(\boldsymbol{w}^\top \boldsymbol{X} - c)_+^p] \right)^{\frac{1}{p}} \right) \right\}$$

$$s.t. \quad \boldsymbol{w}^\top \boldsymbol{e} = 1.$$

Robust SMCR-targeted portfolio selection

Proposition 4

(1) If $\mathcal{M} = \mathcal{M}(\boldsymbol{\mu}, \Gamma)$ and $\frac{(\lambda+1)^2}{\lambda^2(\theta^2-1)} < v_0$, we have the optimal portfolio $\boldsymbol{w}^*$

$$\boldsymbol{w}^* = (\Gamma^{-1}\boldsymbol{\mu}, \quad \Gamma^{-1}\boldsymbol{e}) \begin{pmatrix} v_0 & -v_1 \\ -v_1 & v_2 \end{pmatrix} \begin{pmatrix} \zeta^* \\ 1 \end{pmatrix}, \tag{1}$$

where

$$\zeta^* = \frac{v_1}{v_0} - \sqrt{\frac{(\lambda+1)^2 (v_0 v_2 - v_1^2)}{\lambda^2(\theta^2-1)v_0 - (\lambda+1)^2}}.$$

(2) If $\mathcal{M} = \mathcal{M}_S(\boldsymbol{\mu}, \Gamma)$, we have the optimal portfolio $\boldsymbol{w}^*$

(i) If $\theta \leq \sqrt{2}$ and $\frac{(\lambda+1)^2}{\lambda^2(\theta^2-1)} < v_0$, the optimal portfolio is $\boldsymbol{w}^*$ as stated in (1).

(ii) If $\theta > \sqrt{2}$ and $\frac{2(\lambda+1)^2}{(\lambda\theta)^2} < v_0$, the optimal portfolio $\boldsymbol{w}^*$ is

$$\boldsymbol{w}^* = (\Gamma^{-1}\boldsymbol{\mu}, \quad \Gamma^{-1}\boldsymbol{e}) \begin{pmatrix} v_0 & -v_1 \\ -v_1 & v_2 \end{pmatrix} \begin{pmatrix} \zeta_S^* \\ 1 \end{pmatrix},$$

Robust SMCR-targeted portfolio selection (cont')

Proposition 4 (cont')

where

$$\zeta_S^* = \frac{v_1}{v_0} - \sqrt{\frac{(\lambda\theta)^2 \left(v_0 v_2 - v_1^2\right)}{(\lambda\theta)^2 v_0 - 2(\lambda+1)^2}}.$$

(3) If $\mathcal{M} = \mathcal{M}_m(\boldsymbol{\mu}, \Gamma)$, then

$$\boldsymbol{w}^* = (\Gamma^{-1}\boldsymbol{\mu}, \quad \Gamma^{-1}\boldsymbol{e}) \begin{pmatrix} v_0 & -v_1 \\ -v_1 & v_2 \end{pmatrix} \begin{pmatrix} \zeta_m^* \\ 1 \end{pmatrix},$$

where $\zeta_m^* = \arg\min_{\zeta \in \mathbb{R}} g_m(\zeta, \sqrt{v_0\zeta^2 - 2v_1\zeta + v_2})$, and $g_m(\mu, \sigma) = \mu + \lambda \min_{c \in \mathbb{R}}\{c + \theta\sqrt{h(c, \mu, \sigma)}\}$ with $h(c, \mu, \sigma) = \sup_{F \in \mathcal{L}_m(\mu,\sigma)} \mathbb{E}^F[(X - c)_+^2]$.

Empirical analysis with financial data

- Yahoo!Finance: Apple Inc. (AAPL), Netflix Inc. (NFLX), Alphabet Inc. (GOOG), and eBay Inc. (EBAY)
- Three-year period daily losses from January 1, 2019, to January 1, 2022 (757 observations)

Stocks	Mean (μ)	Covariance matrix (Γ)			
AAPL	-0.0021	0.00050589	0.00028480	0.00020685	0.00029607
NFLX	-0.0014	0.00028480	0.00036718	0.00015550	0.00023912
GOOG	-0.0010	0.00020685	0.00015550	0.00040873	0.00013994
EBAY	-0.0011	0.00029607	0.00023912	0.00013994	0.00037593

Table: Summary of four selected stocks mean and covariance matrix.

Robust TSV-targeted portfolio

	$t=-0.1$				
	$m=0.001$	$m=0.005$	$m=0.01$	$m=0.05$	$m=0.1$
AAPL	-0.05722	-0.05005	-0.04408	-0.00888	0.35129
NFLX	0.32198	0.32190	0.32184	0.32148	0.31777
GOOG	0.38681	0.38441	0.38241	0.37062	0.24994
EBAY	0.34843	0.34374	0.33983	0.31679	0.08101
	$t=-0.5$				
AAPL	-0.05831	-0.05477	-0.05204	-0.03985	-0.02931
NFLX	0.32199	0.32195	0.32192	0.32180	0.32169
GOOG	0.38717	0.38599	0.38508	0.38099	0.37746
EBAY	0.34914	0.34683	0.34504	0.33706	0.33016
	$t=-1$				
AAPL	-0.05876	-0.05623	-0.05431	-0.04594	-0.03922
NFLX	0.32199	0.32197	0.32195	0.32186	0.32179
GOOG	0.38733	0.38648	0.38584	0.38303	0.38078
EBAY	0.34944	0.34779	0.34653	0.34105	0.33665

Table: The optimal robust portfolio for the TSV-targeted case when the uncertainty set is induced by $\mathcal{M}_m(\boldsymbol{\mu}, \Gamma)$.

Robust SMCR-targeted portfolio

	$\theta = 20,\ \lambda = 0.5$				
	$m = 0.01$	$m = 0.05$	$m = 0.1$	$m = 0.5$	$m = 1$
AAPL	-0.04712	-0.04684	-0.04665	-0.04583	-0.04522
NFLX	0.32187	0.32187	0.32187	0.32186	0.32185
GOOG	0.38343	0.38333	0.38327	0.38299	0.38279
EBAY	0.34182	0.34164	0.34151	0.34098	0.34057

	$m = 0.01,\ \lambda = 0.5$				
	$\theta = 3$	$\theta = 5$	$\theta = 10$	$\theta = 20$	$\theta = 50$
AAPL	0.02722	-0.00787	-0.03405	-0.04712	-0.05495
NFLX	0.32111	0.32147	0.32174	0.32187	0.32195
GOOG	0.35852	0.37028	0.37905	0.38343	0.38605
EBAY	0.29316	0.31613	0.33326	0.34182	0.34695

Table: The optimal robust portfolio for the SMCR-targeted case when the uncertainty set is induced by $\mathcal{M}_m(\boldsymbol{\mu}, \Gamma)$.

Robust SMCR-targeted portfolio (cont')

	$m = 0.01,\ \theta = 20$				
	$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 10$
AAPL	0.07155	0.00526	-0.04712	-0.05365	-0.05952
NFLX	0.32065	0.32133	0.32187	0.32194	0.32200
GOOG	0.34367	0.36588	0.38343	0.38561	0.38758
EBAY	0.26414	0.30753	0.34182	0.34609	0.34994

Table: The optimal robust portfolio for the SMCR-targeted case when the uncertainty set is induced by $\mathcal{M}_m(\boldsymbol{\mu}, \Gamma)$.

Conclusion

- **One-dimensional**: Derived closed-form worst-case first-order UPM and worst-case target semi-variance under various uncertainty sets including
  - symmetrical distribution,
  - non-negative random loss,
  - constraint on expected losses over target level.

  Main idea: reduce to a corresponding finite-point discrete uncertainty sets.

- **Multi-dimensional**: robust portfolio selection with TSV-targeted and SMCR-targeted objectives.
  Main idea: reduce inner multi-dimensional problem to one-dimensional problem.

- Empirical study using real financial data $\Rightarrow$ other insurance applications?

Reference I

Chen, L., He, S. and Zhang, S. (2011). Tight bounds for some risk measures, with applications to robust portfolio selection. *Operations Research*, **59**(4), 847–865.

Bawa, V. S. (1975). Optimal rules for ordering uncertain prospects. *Journal of Financial Economics*, **2**(1), 95–121.

Ben-Tal, A. and Nemirovski, A. (1998). Robust convex optimization. *Mathematics of operations research*, **23**(4), 769–805.

Bertsimas, D. and Popescu, I. (2002). On the relation between option and stock prices: a convex optimization approach. *Operations Research*, **50**(2), 358–374.

Cai, J. and Tan, K. S. (2007). Optimal retention for a stop-loss reinsurance under the VaR and CTE risk measures. *ASTIN Bulletin*, **37**(1), 93–112.

Cai, J. and Chi, Y. (2020). Optimal reinsurance designs based on risk measures: A review. *Statistical Theory and Related Fields*, **4**(1), 1–13.

Cai, J., Liu, F., Yin, M. (2023). Worst-case risk measures of stop-Loss and limited loss random variables under distribution uncertainty with applications to robust reinsurance.

Gomez, F., Tang, Q. and Tong, Z. (2022). The gradient allocation principle based on the higher moment risk measure.*Journal of Banking and Finance*, **143**, 106544.

Jagannathan, R. (1977). Technical note–Minimax procedure for a class of linear programs under uncertainty. *Operations Research*. **25**(1), 173-177.

Reference II

Kaluszka, M. (2005). Optimal reinsurance under convex principles of premium calculation. *Insurance: Mathematics and Economics*. **36**(3), 375–398.

Liu, H. and Mao, T. (2022). Distributionally robust reinsurance with Value-at-Risk and Conditional Value-at-Risk. *Insurance: Mathematics and Economics*, **107**, 393–417.

Liu, W., Yang, L. and Yu, B. (2021). KDE distributionally robust portfolio optimization with higher moment coherent risk. *Annals of Operations Research*, **307**(1–2), 363–397.

Thank you!